MAGNETOSTATIC FIELDS COMPUTED USING AN INTEGRAL EQUATION DERIVED FROM GREEN'S THEOREMS[§].

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ABSTRACT

A method of computing magnetostatic fields is described that is based on a numerical solution of the integral equation obtained from Green's Theorems. The magnetic scalar potential and its normal derivative on the surfaces of volumes are found by solving a set of linear equations. These are obtained from Green's Second Theorem and the continuity conditions at interfaces between volumes. Results from a two-dimensional computer program are presented and these show the method to be accurate and efficient.

1. INTRODUCTION

The present generation of computer programs for calculating magnetostatic fields in three dimensions are expensive to use and they will continue to be until new algorithms are developed. Changes in computer hardware, e.g. parallel processors, may make it possible to obtain solutions more quickly, but, it is doubtful whether the amount of storage available will change significantly. In this paper the numerical solution of an integral equation derived from Green's Theorems is shown to have many advantages over existing integral equation methods.

Integral equation methods are now widely accepted and the Rutherford Laboratory program GFUN3D [1], which solves the integral equation for the volume distribution of induced magnetisation, has been successfully used for the design of many magnets. As an example of the accuracy of this program the measured and computed results for the homogeneity of an essentially two-dimensional C shaped dipole magnet are shown in Figure I. This accuracy (better than 1 part in 10^4) was obtained by using 10 minutes of CPU time on an IBM 360/195, a cruder model capable of 1% accuracy would typically require 10 seconds CPU time. In the case of strongly three dimensional magnets however, 60 minutes of CPU time are probably required for accuracy better than 1%. Furthermore, for complex problems even when the magnetisation distribution has been computed, the time taken to compute fields at particular points is not trivial. Iselin [2] has proposed a scalar potential method that may prove to be more efficient than GFUN3D which uses the three component magnetisation. An alternative approach is the Boundary Integral Method; this method is based on the numerical solution of an integral equation for the magnetic scalar potential, derived from Green's Theorems. This approach has already been used for the solution of linear flow and elasticity problems [3, 4, 5].

For linear problems, i.e. constant permeability, it is only necessary to define the boundaries of regions with different permeability, together with a far field boundary condition - however the far field boundary can be expanded to infinity. A region may consist of several surfaces that do not touch or intersect and this fact together with the

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use of symmetry allows the calculation of fields with minimal effort. In an appendix an extension is discussed that will make it possible to include non-linear permeability.

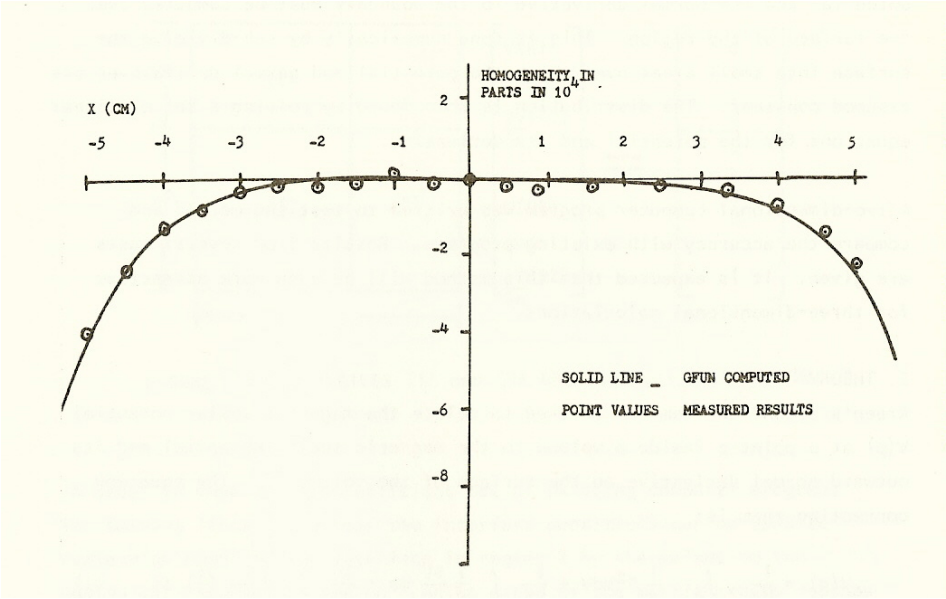

FIGURE 1 - Measured and computed homogeneity of the field produced by a c-shaped magnet with small pole tip shims

To determine the magnetic field distribution in a region the magnetic scalar potential and its normal derivative to the boundary must be computed over the surface of the region. This is done numerically by sub-dividing the surface into small areas over which the potential and normal derivatives are assumed constant. The distribution is then found by solving a set of linear equations for the potential and its derivative. A two-dimensional computer program was written to test the method and compare the accuracy with existing programs. Results from several tests are given. It is expected that this method will be even more attractive for three-dimensional calculations.

2. THEORY

Green's second theorem can be used to relate the magnetic scalar potential *V(p)* at a point *p* inside a volume to the magnetic scalar potential and its outward normal derivative on the surface of the volume [6]. The equation connecting them is:

$$V(p) = \frac{1}{4\pi}\int_v \frac{1}{r}\nabla^2 V dv + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial V}{\partial n} dS - \frac{1}{4\pi}\int_S V \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS \qquad (1)$$

where *r* is the distance between the point *p* and an element of the volume or surface of the region. If the permeability of a region is constant then:

$$\nabla^2 V = 0$$

and therefore the first integral in equation (1) is zero. In Appendix 1 the use of a perturbation term based on the volume integral is shown as a possible means of extending the method to non-linear permeability. If a surface is defined just inside the boundary of a volume and this surface is subdivided into small areas over which V and $\partial V / \partial n$ are constant then equation (1) becomes:

$$V(p) = \frac{1}{4\pi} \sum_{j=1,m} \left(\frac{dV_j}{\partial n_j} \int_{S_j} \frac{1}{r} dS_j - V_j \int_{S_j} \frac{\partial}{\partial n_j} \left(\frac{1}{r} \right) dS_j \right) \tag{2}$$

where the surface is subdivided into *m* area elements. Equation (2) can also be used to express the potential of a point on this surface as a function of the potential and its outward normal derivative on each surface area element.

If the geometric factors relating to the potential and its derivative on every element of the surface are calculated for points at the centroid of every area element, then providing *V* or $\partial V / \partial n$ is known on every area element, the unknown values can be found by solving a set of exactly determined linear equations.

Of more interest is a problem consisting of regions with different permeability where there are interfaces between the regions. For example, consider a two-region problem, where region 1 has permeability μ_1 and region 2 has permeability μ_2 (This could correspond to region 1 being iron and region 2 air). There must be some driving field; however this is at present of no account except that a distribution of field H_v is assumed to be produced by a set of current carrying conductors. A surface is defined just inside each region and this surface is subdivided into small elements with an exact correspondence between the elements across the interface between the regions.

Equation (2) then gives for each surface element:

$$V_j(R_1) - \frac{1}{4\pi} \sum_{i=1,n1} \left(\frac{dV_i(R_1)}{\partial n_i} \int_{S_i} \frac{1}{r} dS_i - V_i(R_1) \int_{S_i} \frac{\partial}{\partial n_i} \left(\frac{1}{r} \right) dS_i \right) = 0 \tag{3}$$

$$V_k(R_2) - \frac{1}{4\pi} \sum_{i=1,n2} \left(\frac{dV_i(R_2)}{\partial n_i} \int_{S_i} \frac{1}{r} dS_i - V_i(R_2) \int_{S_i} \frac{\partial}{\partial n_i} \left(\frac{1}{r} \right) dS_i \right) = 0 \tag{4}$$

where $V_j(R1)$ are the potentials in region 1 and $V_k(R2)$ are potentials in region 2. On the interface between the two regions *V* and $\partial V / \partial n$ on the surface elements are unknown in both regions. If *V* or $\partial V / \partial n$ is known on the surface elements that are not on the interface then the set of linear equations formed from (3) and (4) will still be under-determined. Two extra equations must be introduced for each interface element and these can be obtained from the interface continuity conditions. The equations are:

$$V_j(R_1) = V_k(R_2) \tag{5}$$

$$\mu_1(-\frac{\partial V_j(R1)}{\partial n_j} + H_{n_j}(R1)) = \mu_2(+\frac{\partial V_k(R2)}{\partial n_k} + H_{n_j}(R1)) \qquad (6)$$

where $H_{n_j}(R1)$ is the outward normal component of the driving field on element *j* of region 1. The same ideas can be applied to problems consisting of any number of regions.

It is interesting at this stage to examine the set of equations generated to determine *V* and $\partial V / \partial n$ in a two region problem, where there is an interface between the regions. A pictorial representation of the equations is shown in Figure 2. There are nl and n2 sides and ml and m2 unknowns in region and 2 respectively. The submatrix (1) is dense and is formed from the coefficients from equation (3) applied to the element of region 1. Similarly submatrix (4) comes from region 2. The submatrices (2) and (3) are sparse (two unknowns per row) and are generated from the interface conditions. The other areas contain zeros. If on the boundary surfaces where the potential or its derivative is known the value is zero then all the right-hand sides are zero except those corresponding to the normal **B** continuous boundary conditions.

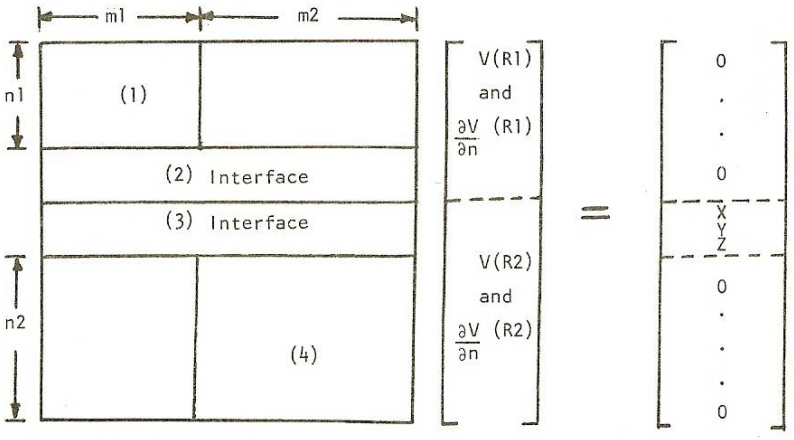

FIGURE 2 - A Representation of the Set of Linear Equations Required to Solve For the Potential and its Normal Derivative in a 2 Region Problem

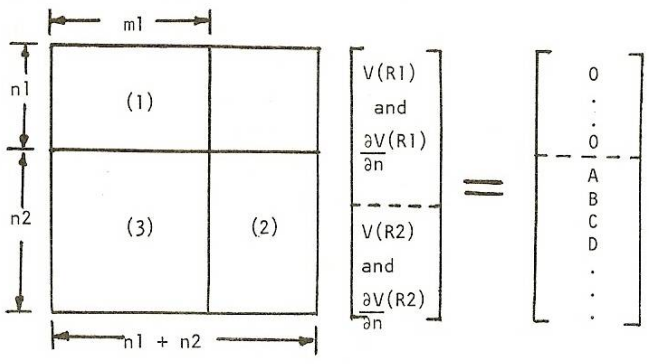

FIGURE 3 - Submatrices (1) and (2) are dense, (3) is sparse

In order to make the most efficient use of existing computer programs for solving linear equations the interface conditions can be used to replace unknowns on the interface in region 2 by the values in the equivalent elements in region 1. The order of the matrix can be reduced using this technique but at the expense of the loss of the blocking that previously existed. From a long term point of view it would be more efficient to use the blocked matrix and special factorising methods. Figure 3 shows the structure of the set of equations after order reduction has taken place. In the case of a problem only consisting of interfaces the order is reduced to half its previous size.

3. SYMMETRY

The number of unknowns in a problem can be reduced significantly when the geometry and its associated potential distribution possesses a known rotational or reflective symmetry. The two methods that can be employed to make use of this symmetry are shown pictorially in Figures 4 and 5. In Figure 4 a model of a dipole magnet is shown where the Dirichlet and Neumann boundary values have been used to imply the rest of the model. In Figure 5 the whole model is shown but, because the potentials in the 2nd, 3rd and 4th quadrants have an exact equivalence to those in the first quadrant, the potentials in the first quadrant are the only ones which must be computed explicitly.

The far field boundary shown in Figure 5 can be expanded to infinity because there are no boundary connections between it and the magnet; the far field boundary then has no effect on the problem whatsoever. This is obvious for real problems where the potential and its normal derivative to the far boundary can be defined as zero. It is not immediately clear in the two-dimensional infinite limit because the potential from a boundary side becomes infinite at large distances. However the divergence of the potential from a complete surface must be zero and therefore the contributions from all elements of a surface will cancel to produce zero potential at infinity.

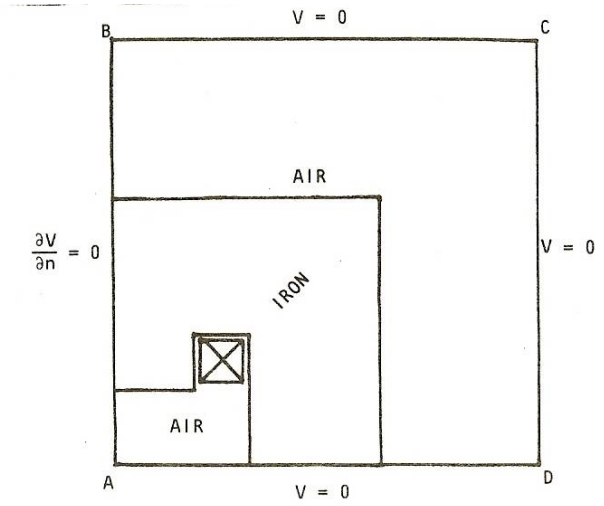

Figure 4 - Boundary integral method model using Neumann and Dirichlet boundary values

4. APPLICATIONS OF THE METHOD

A two-dimensional magnetostatic computer program was written to test the accuracy and efficiency of the method. The results for the program were very encouraging. In the program the fields from infinitely long conductors with polygonal cross section and curvilinear faces were computed using existing analytic expressions [7]. The boundaries between regions of different permeabilities were subdivided into plane faces over which the potential and its normal derivative were assumed to be constant. The expression for the potential and field from such faces are given in Appendix 2. The integrals can be evaluated for higher order basis functions but this leads to problems at external corners because the integrals have singular kernels. This problem can be solved but it was simply avoided in the present program by computing the potentials at the centroid of each element where the integral is well behaved. The program can be run interactively on. the Rutherford Laboratory. IBM 360/195 and in this version an elegant data input package was used for specifying the boundary data of polyhedra [8].

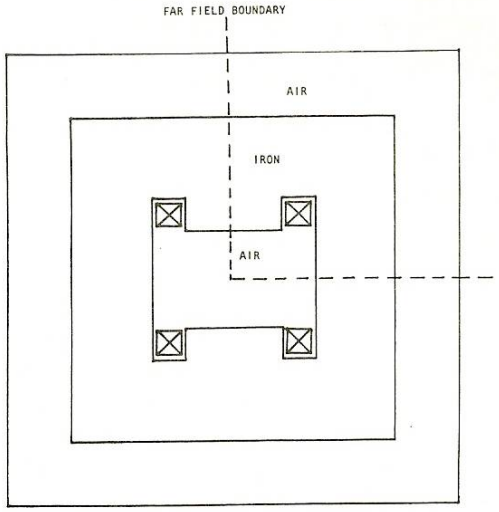

FIGURE 5 - Boundary integral method model using symmetry - the far field boundary is infinity

5. RESULTS

The results from two test cases are included in this section; a comparison of analytic and computed results for the field in a hollow, infinitely long, constant permeability cylinder in a uniform external field; and a comparison of the GFUN and Boundary Integral Method computed fields for a two-dimensional C magnet.

(a) **Hollow Cylinder**. The fields in a hollow infinitely long constant permeability cylinder in a uniform field perpendicular to the axis of the cylinder were computed using the Boundary Integral Method. The inside radius of the cylinder was 5 cms and the outside radius 10 cms. The cylinder was approximated by many-sided polyhedra and symmetry was used so that only potentials and derivatives in the first quadrant were computed explicitly. In Figures 6 and 7 the computed shielding factor of the cylinder is plotted as a function of the number of boundary faces for cylinders with relative permeabilities of 100 and 1000. The accuracy is very good, and most of the

error is due to the polygonal approximation. The field in the hollow centre should be uniform and in the computed cases the homogeneity was always better than 2 in 10^4. An interesting point to note about the results is that the fields at points inside the cylinder were obtained as accurately as the shielding factor; this is not true in the GFUN program where eigenvalue solutions can be obtained.

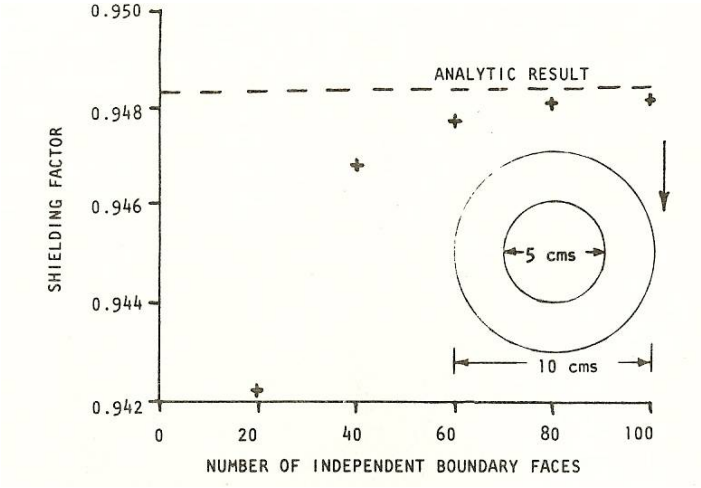

FIGURE 6 - Shielding factor of hollow ferromagnetic cylinder – Inside Radius 5 cm. Outside radius 10 cm. permeability 100 - as a function of the number of independent boundary faces in the model.

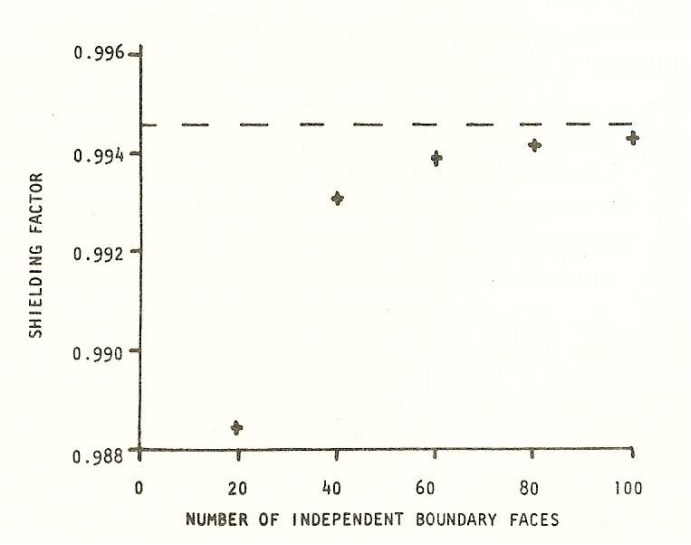

FIGURE 7 - Shielding factor of a hollow ferromagnetic cylinder – dimensions as figure 6, permeability 1000 - as a function of the number of independent boundary faces.

(b) **C-Shaped Dipole Magnet** The geometry of this magnet is shown in Figures 8 and 9, Figure 8 shows the GFUN model and Figure 9 the Boundary Integral Method model. The results in Figure 1 have shown that GFUN gives accuracies of the order of 0.01% for the homogeneity of this type of C magnet. GFUN was therefore used to compute the field homogeneity of the magnet shown in Figure 8 for steel with a relative permeability of 1000.0. In Figure 10 the GFUN results are compared to those

obtained using the Boundary. Integral Method (BIM) for several different models. Symmetry was used and therefore only the upper Y plane was computed explicitly. (In both these cases the far field boundary was at infinity.) The results for this case are again good. Figure 11 shows a computed map of lines of constant scalar potential for the 140 element BIM model.

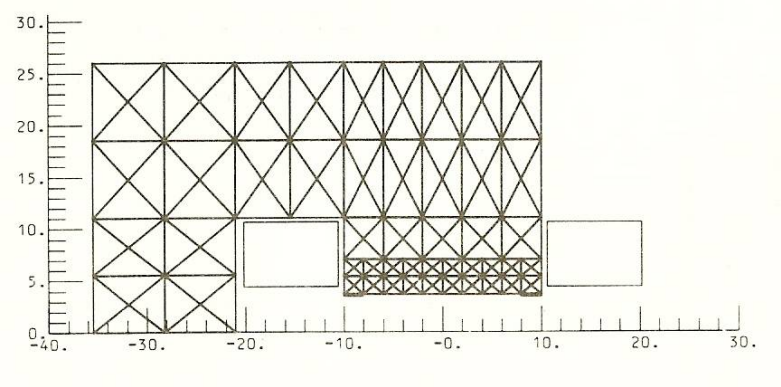

FIGURE 8 - GFUN Model of a two-dimensional C-magnet

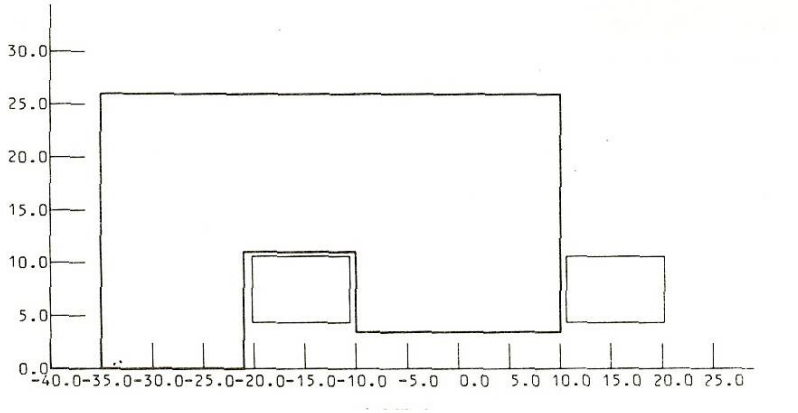

FIGURE 8 - BIM Model of a two-dimensional C-magnet

6. CONCLUSIONS

The results achieved for two-dimensional magnet problems are encouraging and appear to be competitive with other methods. The extension of the Boundary Integral Formulation to three dimensions is relatively straightforward and should in principle lead to a more efficient algorithm than the one currently in use in GFUN. For example, the table I compares predicted computing time (seconds) for a range of problems, i.e. for existing GFUN, BIM and the Scalar Potential Integral Equation [2] formulation. The table also gives times for computing a single field point.

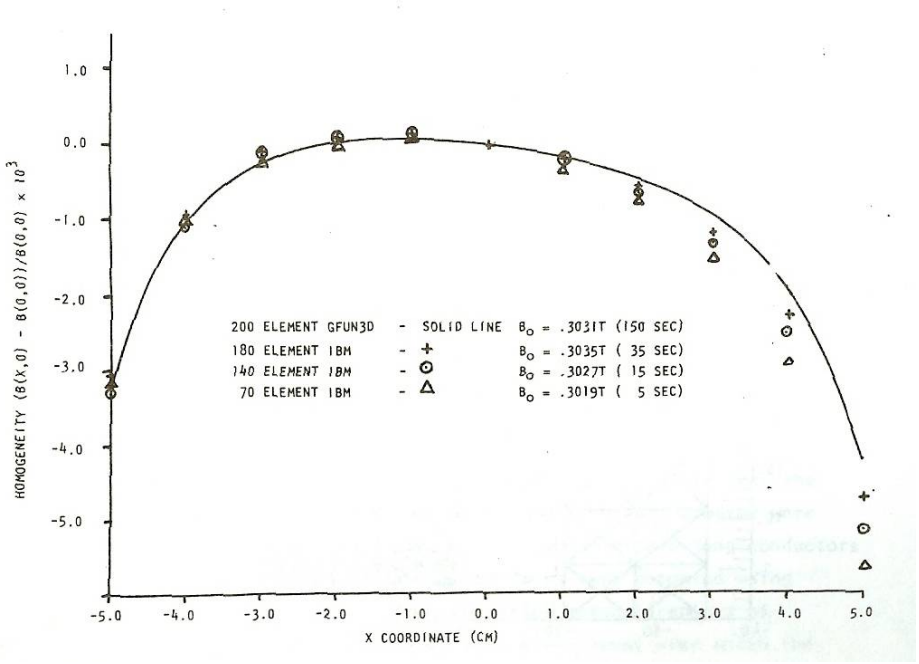

FIGURE 10 - Computed homogeneity of the field under the pole tip of the C- magnet shown in figure 8.

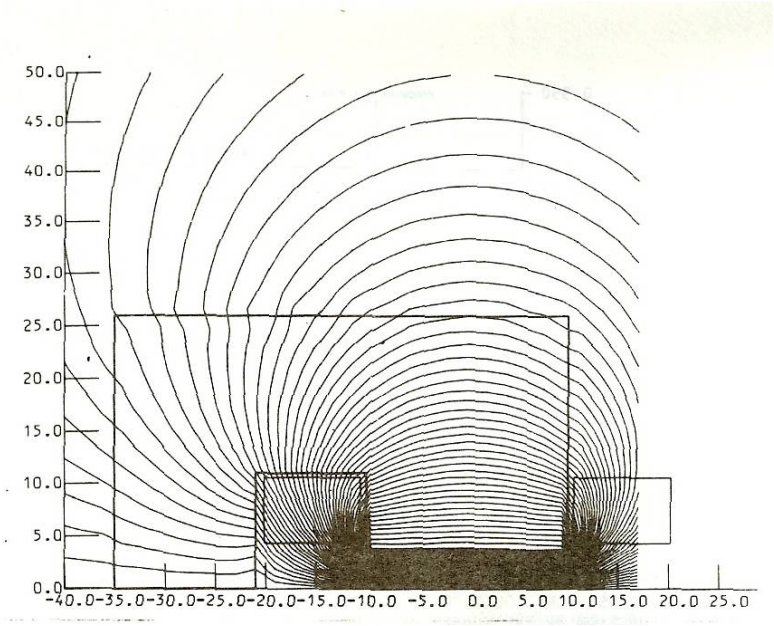

FIGURE 11 - A map of the computed magnetic scalar potential for a C-magnet

Table I: Computing Times

Volume Elements	Surface Elements in BIM	GFUN Magnetisation method		BIM		Scalar Int. Eq.	
		Int. Eq.	Single Field Point	Int. Eq.	Single Field Point	Int. Eq.	Single Field Point
216	216	114	2.16	12	0.36	4.2	2.16
343	294	450	3.43	30	0.49	18.0	3.43
512	384	1488	5.12	70	0.64	55.0	5.12
730	486	4320	7.30	133	0.80	162.0	7.30

It can be seen that, as the number of elements increases, BIM compares very favourably with the Scalar Integral Equation method both for the main solution and for fields at single points. Since the existing program is restricted to constant permeability problems the best method for solving the non-linear problems must be established - the multi-region option outlined in Appendix I Section 2 will be tried first by modifying the existing two-dimensional program.

Finally, it should be emphasised that this method has a far wider range of applicably than Magnetostatics for example, solution of current flow potentials in association with eddy currents [8]; also it may be used to advantage in improving the efficiencies of programs already developed such as GFUN for computing the fields at single points.

7. ACKNOWLEDGEMENTS

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APPENDIX 1

EXTENSION OF THE GREEN'S THEOREM APPROACH TO NON-LINEAR MAGNETOSTATIC PROBLEMS.

There are two possible methods of extending the method to cover non linear problems; the first involves using a perturbation term based on the volume integral in equation (1); the second would require the whole of an iron volume to be subdivided into separate volume elements on the surfaces of which the potential and its normal derivative are computed.

(1) **Perturbation term method**. The magnetic field $\mathbf{H_T}$ at a point can be divided into two parts - $\mathbf{H_C}$ due to currents and $\mathbf{H_M}$ due to the iron.

$$\mathbf{H}_T = \mathbf{H}_C + \mathbf{H}_M \tag{7}$$

Since Div **B** = 0, then:

$$\mathrm{Div}\left(\mu\mathbf{H}_C + \mu\mathbf{H}_M\right) = 0 \tag{8}$$

(only isotropic materials are considered here)

From equation (8)

$$\mathrm{Div}\left(\mu\mathrm{H}\right) = \left(\nabla\mu\right).\mathbf{H}_M + \mu\left(\nabla.\mathbf{H}_M\right) = \left(\nabla\mu\right).\mathbf{H}_M - \mu\nabla^2 V = -\mathrm{Div}\left(\mu\mathbf{H}_C\right)$$

Since:

$$\mathrm{Div}(\mu\mathbf{H}_C) = 0$$

$$\nabla^2 V = \frac{1}{\mu}\nabla\mu.\mathbf{H}_M \tag{9}$$

Combining equations (2) and (9):

$$4\pi V(p) = \int_v \frac{1}{\mu}\left(\nabla\mu.\mathbf{H}_M\right)\frac{dv}{r} + \int_S \frac{1}{r}\frac{\partial V}{\partial n}dS - \int_S V\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{10}$$

This equation could be solved numerically by calculating the contributions of the volume integral when the solution for V and $\partial V/\partial n$ is known. Using a simple iterative scheme the values of V and $\partial V/\partial n$ could then be updated by resolving

equations (3) and (4) with the volume integral contribution added to the right-hand sides and the continuity conditions modified.

(2) **Volume subdivision method**. The existing two-dimensional program can be used to evaluate this method. The ferromagnetic regions of a problem must be divided into small elements over which the change in permeability is small. The equations to be solved are unchanged but an iterative method must be used to converge the solutions for the permeabilities. This method has several advantages; the matrix to be solved is banded and sparse and has a similar structure to those obtained in finite element methods; a numerical calculation of the gradient of μ is not needed. It is hoped to try this second method if present improvements to integral equation methods[2] do not fulfil their promise.

APPENDIX 2

EXPRESSIONS FOR THE FIELD AND POTENTIAL FROM SINGLE AND DOUBLE LAYER SURFACE CHARGES ON INFINITELY LONG PLANE FACES OF FINITE WIDTH

A typical region consisting of many boundary faces is shown in Figure 12. All the expressions given below are for points in the local coordinate system of a boundary face - Figure 13. The faces are infinitely long and of the plane of the paper - in the Z direction:

(1) **Potentials** The integrals to be evaluated are shown in equation (2).

The potential at a point *p(x,y)* is:

$$V(p) = \frac{1}{4\pi}\left[V_j(2\theta) + 2\frac{\partial V_j (x\ln(r_1/r_2) + b\ln(r_1/r_2) - 2b + y\theta)}{\partial n} \right]$$

where 2b is the width of the face.

(2) **Fields**. The field at point *p(x,y)* is:

$\mathbf{H} = -\text{grad}\, V(p)$

Therefore:

$$H_x = \frac{1}{2\pi}[V_j y\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + \frac{\partial V_j}{\partial n}\{x(\frac{(x+b)}{r_1^2} - \frac{(x-b)}{r_2^2}) + \ln(r_1/r_2) + b(\frac{(x+b)}{r_1^2} - \frac{(x-b)}{r_2^2}) + (\frac{y^2}{r_1^2} - \frac{y^2}{r_2^2})\}]$$

$$H_y = \frac{1}{2\pi}[V_j\left(\frac{(x-b)}{r_1^2} - \frac{(x-b)}{r_2^2}\right) + \frac{\partial V_j}{\partial n}\{x(\frac{y}{r_1^2} - \frac{y}{r_2^2}) + y(\frac{y}{r_1^2} - \frac{y}{r_2^2}) + b(\frac{(x-b)}{r_2^2} - \frac{(x+b)}{r_1^2}) + \theta\}]$$

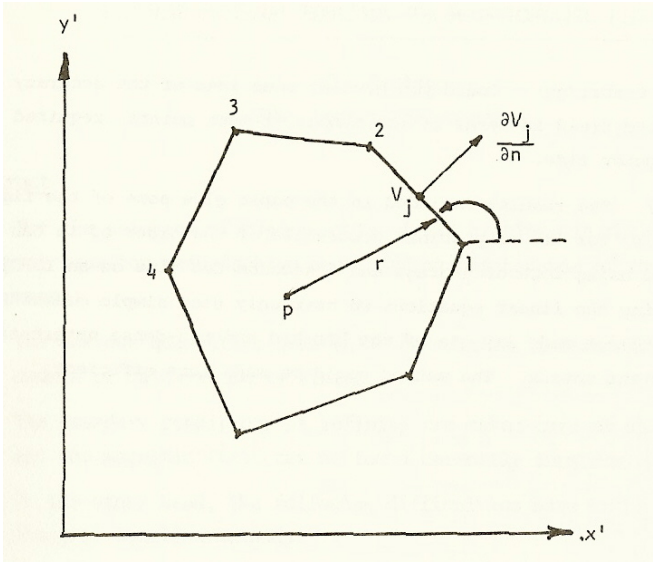

FIGURE 12 - Global coordinate system
A boundary surface subdivided into elements

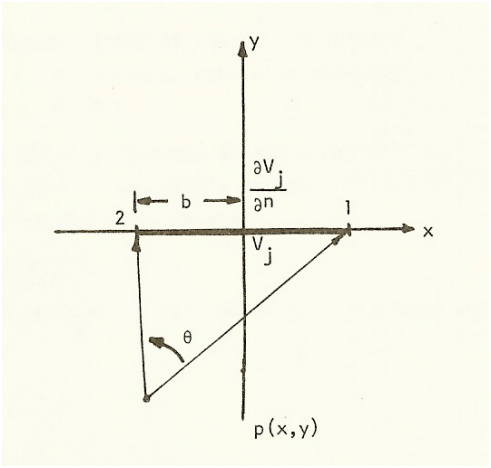

FIGURE 13 - Local coordinate system of an element

Addendum 9 May, 200

As conference proceedings are now difficult to obtain this paper has been copied without any substantive changes, other than those needed to reformat the text. The original computer output has been retained throughout and only very minor corrections to the text have been made.

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